On Polyakov's basic variational formula for loop spaces

Ioannis P. Zois[1]

Mathematical Institute,
24-29 St. Giles', Oxford OX1 3LB

[1]izois@maths.ox.ac.uk

Abstract

We use the homological algebra context to give a more rigorous proof of Polyakov's basic variational formula for loop spaces.
PACS classification: 11.10.-z, 11.15.-q, 11.30.Ly

0.1 Introduction-Motivation

It is known for some time now that one can reformulate Yang-Mills theory as *non-linear σ model* (abriviated to *"nlσm"* in the sequel) on the *loop space* [9]. This was related originally to the problem of *confinement of quarks* [9], [10]. In addition recently a fascinating "electric-magnetic" duality was observed in the loop space formulation of Yang-Mills theories [4].

Apart from the obvious disadvantages one has when formulating a field theory (in particular nlσm here) on an infinite dimensional space (namely the loop space of the initial manifold), there are however some simplifications in field equations and an extra $U(1)$ symmetry (coming from rotating loops) and hence this approach is not merely an extra undesired nuance [9], [4]. There are also some mathematical advantages related to the Duistermat-Heckman formula [2], [11] and to the heat equation proof of the Atiyah-Singer index theorem [3]. This reformulation is based on Polyakov's basic variational formula [9]:

$$\delta h(c) = \int_0^1 dsP\left(exp\int_0^s A_\mu dx^\mu\right)F_{\mu\nu}(c(s))\frac{dx^\nu(s)}{ds}P\left(exp\int_s^1 A_\mu dx^\mu\right)\delta x^\mu(s)$$

where F is the curvature of a connection 1-form A on spacetime, c is a loop, $h(c)$ is the holonomy element:

$$h(c) = Pexp\int_c A$$

and P is the well-known *Dyson ordering*. The loop c is described via the function $x^\mu(s)$.

Using the isomorphism induced by the *iterated integral map* between the *Hochschild homology* of the associative algebra of differential forms of the original manifold and the *de Rham cohomology* of the corresponding loop space when the manifold is simply connected, (see [6], [7], [5]), we give a more mathematically rigorous proof of Polyakov's result.

We hope moreover that some ideas and techniques from iterated integrals will be of some use to the canonical quantization of gravity using Astekhar variables because loop spaces are also important in that context.

In more concrete terms, let M be a four dimensional simply connected manifold (assumed to be spacetime) and let LM be its loop space, namely the set of smooth maps from the circle T to M. The dimension of M is not crucial, it can be anything, we choose four due to physical significance. What is crucial is that M has to be *simply connected*. In quantum field theory context usualy spacetime has the topology of $\mathbf{R}^4$ with either Euclidian or Minkowski metric. We denote by $\Omega(M)$ the associative algebra of differential forms on M. Then the above mentioned result simply states that if M is simply connected then the Hochschild homology of $\Omega(M)$ is isomorphic to the de Rham cohomology of LM. The isomorphism is the one induced by the iterated integral map. As a reference for Hochschild homology, see [8].

We organise this paper as follows: In section 2 we briefly present Polyakov's main ideas; in section 3 we give a brief review of loop spaces. Since iterated integrals are not, we think, *lingua franca* in physics, we give in section 4 the basic definitions. After that we give the proof itself in section 5. Then we end up with some remarks in section 6.

0.2 Yang-Mills theory as a nlσm on Loop space

(The main reference in this section is [9]).

In establishing the above formulation, the basic role is played by a well-known object, the element of the holonomy group. Following standard physics terminology we write the holonomy element h as

$$h(c) = Pexp \int_c A$$

where c is some loop, A is a connection 1-form and P stands for the *Dyson ordering* along this loop. We now consider h as a chiral field. In mathematical language h is a zero form on the loop space of M with values in the group G. The underlying mathematical structure is a principal bundle X over M with structure Lie group G assumed to be compact and connected. We introduce a connection $\tilde{A}$ on the loop space by the formula

$$\tilde{A}_\mu(s,c) := \frac{\delta h}{\delta x^\mu(s)} h^{-1}$$

where

$$\frac{\delta}{\delta x^\mu(s)}$$

is the loop derivative [4].

In the above expression the loop c is parametrised by the function $x^\mu(s)$ and clearly the index μ takes the values 0,...,3. The above defined connection should be reparametrisation invariant because $h(c)$ is and hence one must have

$$\frac{dx^\mu(s)}{ds} \tilde{A}_\mu(s,c) = 0$$

From the definition of the connection on the loop space one can deduce that this connection is flat [9], namely

$$\frac{\delta \tilde{A}_\mu(s,c)}{\delta x^\nu(s_1)} - \frac{\delta \tilde{A}_\nu(s_1,c)}{\delta x^\mu(s)} + [\tilde{A}_\mu(s,c), \tilde{A}_\nu(s_1,c)] = 0$$

We would like to note here that strictly speaking this is not a connection on the G-bundle over the loop space since it is defined explicitly in terms of the

given holonomy and there is no notion of "gauge transformation" on $\tilde{A}_\mu(s,c)$ itself. However for convenience we shall refer to it as a connection.

The important result is that the Yang-Mills equations take a simple form in terms of the connection on the loop space [9], namely

$$\frac{\delta \tilde{A}_\mu(s,c)}{\delta x^\mu(s)} = 0$$

So to sum up, one has two important results when formulating Yang-Mills equations on loop spaces:

1. The connection 1-form on the loop space is *flat* even if the space-time connection one starts with is not.

2. The Yang-Mills equations reduce to a "divergenceless-like" condition for the connection 1-form on loop space. (Actually the word "like" is very important, one cannot define a Hodge star on the loop space forms since the de Rham complex is not bounded above—due to infinite dimensionality—although one has a naturally induced metric on the loop space if the manifold itself has a metric, see below).

These two results mentioned above are based on the following variational formula:

$$\delta h(c) = \int_0^1 dsP\left(exp\int_0^s A_\mu dx^\mu\right) F_{\mu\nu}(c(s))\frac{dx^\nu(s)}{ds}P\left(exp\int_s^1 A_\mu dx^\mu\right)\delta x^\mu(s)$$

where F is the curvature of the connection A on space time. Our proof explains the appearence of the curvature in this formula quite naturally.

0.3 Loop spaces

In this section we review some well-known facts about loop spaces in general. More details can be found in [2].

Consider a finite dimensional compact orientable Riemannian manifold M. Then by definition the loop space of M is the following infinite dimensional manifold

$$LM := Map(T, M)$$

consisting of all smooth maps from the circle T to our manifold. Our description of LM will not be absolutelly rigorous, we ignore analytic issues.

Thus a point on LM is by definition a smooth map $\phi : T \to M$ and the tangent space $T_\phi LM$ of LM at ϕ can be identified with the space of sections of the vector bundle $\phi^*(TM)$, the tangent bundle TM of M pulled-back to T by ϕ.

The metric on M defines a metric on $\phi^*(TM)$ and hence by integration over T we get an inner product on the space of sections. This defines a pre-Hilbert space structure on $T_\phi LM$. Next we introduce the Levi-Civita connection ∇ on M. This induces a connection on the bundle $\phi^*(TM)$ and hence (evaluating along the tangent vector to the loop) a covariant derivative operator ∇_ϕ. This is a skew-adjoint operator on the space of sections $T_\phi LM$ and hence, using the inner product, it defines a skew-bilinear form on $T_\phi LM$. As we now vary the point $\phi \in LM$ we get a 2-form on LM. One can prove that it is closed, the proof based crucially on the use of the Levi-Civita connection on M. However this form is not non-degenerate in general, this is so only at the points ϕ for which ∇_ϕ has a zero eigenvalue, i.e. a tangent vector to M which is covariantly constant along the loop ϕ. The Hamiltonian function H associated to the obvious action of the circle can nonetheless still be defined as follows: recall that the energy E of a loop ϕ is defined as

$$E(\phi) = \frac{1}{2}\int_T |d\phi|$$

Computing the derivative of E in the direction of a tangent vector $\xi \in T_\phi$ we get

$$(dE, \xi) = \int_T < \frac{d\phi}{dt}, \nabla_\phi \xi >$$

which establishes that $E = H$. This allows one to get an analogue of the Duistermaat-Heckman formula in infinite dimensions.

The lesson here therefore is that the loop space of any Riemannian manifold is *almost* a symplectic manifold and in fact most of the "symplectic" things can be done on the loop space (if one can overcome the infinite dimensions!).

The orientability of LM can be understood as follows: we have the natural evaluation map $f : T \times LM \to M$; pulling back by f^* and then integrating over T induces a homomorphism

$$a : H^2(M; \mathbf{Z}_2) \to H^1(LM; \mathbf{Z}_2)$$

The image of the second Stiefel-Whitney class of M is then the obstruction to orientability of LM. In particular, if M is spin, then LM is orientable. The converse holds if M is also simply connected.

We would like to mention another fact which is not relevant for our immediate discusion but it is useful to know and it is actually one of the main motivations to study loop spaces in general: the Wiener integration on the loop space LM is related to the heat equation on M, thus giving, another way to calculate the index of elliptic operators on M using data from LM. One however must be careful to distinguish between the Wiener measure (using Riemannian structure) and the Liouville meassure (using symplectic structure) in the case of LM. They

are related via the Pfaffian. This fact is also useful in physics, in SUSY nlσm.(cf [11]).

0.4 Iterated integrals

We start with some motivation first. Consider the following ODE

$$\frac{d\phi(t)}{dt} = a(t)\phi(t)$$

where $a(t)$ is a given function and we want to solve this in the interval [0,1] given the initial condition $\phi(0) = 1$. This is trivial. Yet one may ask the following non trivial question: is it possible to calculate the single *value* $\phi(1)$ *without* solving the equation with respect to the function $\phi(t)$?

The answer is *yes* and the formula is the following:

$$\phi(1) = \sum_{k=0}^{\infty} \int_{\Delta_k} a(t_1)...a(t_k)dt_1...dt_k$$

where Δ_k is the standard k-simplex

$$\{(t_1, ..., t_k) \in \mathbf{R}^k : 0 \leq t_1 \leq ... \leq t_k \leq 1\}$$

The above is a sum of iterated integrals, namely each term k *is* an iterated integral.

One can easily notice that the above equation suitably generalised, actually gives the correct expression for parallel transport of a vector field (replacing ϕ above) given a connection 1-form A (replacing $a(t)$ above), namely essentially the covariant derivative. And now we think the whole thing starts to take shape. In the above setting then, assuming [0,1] parametrising a circle, the value $\phi(1)$ is exactly the holonomy (more precisely the final vector which is the holonomy times the initial vector as matrices). This is actually the observation that made us think about relating Polyakov's formulae on loop spaces with iterated integrals. Let us however, before giving formal definitions, write down an iterated integral explicitly: suppose we are in Euclidian space $\mathbf{R^n}$. If $w = w_i(x)dx^i$ and $v = v_i(x)dx^i$ are two real valued 1-forms on $\mathbf{R^n}$ and suppose $a : [0,1] \rightarrow \mathbf{R^n}$ is a path, then the "twice" iterated integral of the forms w and v is by definition the following expression:

$$\int_0^1 [\int_0^t w_i(x(a(t)))dx^i(a(t))]v_j(x(a(t)))dx^j(a(t))$$

We now pass directly to the definitions on the loop spaces. We shall begin with some general facts about spaces carrying smooth circle actions. (We write

them specifically for the loop spaces, but they hold in general for spaces carrying smooth circle actions).

The loop space LM may be given the structure of an infinite dimensional manifold modeled on a Frechet space. The circle group acts smoothly by rotating loops, namely $(\phi_t c)(s) = c(s+t)$ where $c(s)$ is a loop and ϕ_t is a smooth 1-parameter group of diffeomorphisms with period 1 which describes the smooth circle action. This natural circle action on LM defines several operators on the space of differential forms on LM. The first is the contraction with the vector field which is tangent to the loop, namely the generator of the T-action, which will be denoted by i. Then there is an averaging operator Θ defined via

$$\Theta(w) = \int_0^1 \phi_t^* w dt$$

Furthermore there is a sequence of operators p_k defined as follows:

$$p_k : \Omega(LM)^{\otimes k} \rightarrow \Omega(LM)$$

where $1 \leq k < \infty$

The explicit formula is the following: given a form w on LM, let $w(t)$ be the form $\phi_t^*(w)$ on LM; then one has:

$$p_k(w_1, ..., w_k) = \int_{\Delta_k} iw_1(t_1) \wedge ... \wedge iw_k(t_k) dt_1 ... dt_k$$

where Δ_k is the standard k-simplex

$$\{(t_1, ..., t_k) \in \mathbf{R}^k | 0 \leq t_1 \leq ... \leq t_k \leq 1\}$$

We shall now give the key property of the maps p_k :

Proposition:

If $\epsilon_i = |w_1| + ... + |w_k| - i$, *then*

$$\begin{gathered} dp_k(w_1, ..., w_k) = -\sum_{i=1}^{k} (-1)^{\epsilon_i - 1} p_k(w_1, ..., dw_i, ... w_k) \\ -w_1 p_{k-1}(w_2, ..., w_k) \\ -\sum_{i=1}^{k-1} (-1)^{\epsilon_i} p_{k-1}(w_1, ..., w_i w_{i+1}, ..., w_k) \\ +(-1)^{\epsilon_{k-1}} p_{k-1}(w_1, ..., w_{k-1}) w_k \end{gathered}$$

This formula simply states the fact that

$$\sum_{k=0}^{\infty} p_k$$

is a Hochschild cocycle on the differential graded algebra (DGA for short) $\Omega(LM)$ with coefficients in $\Omega(LM)$ itself.

The proof of the above Proposition is by direct computation using the explicit formula we gave for the maps p_k above (cf [6]).

One can observe that p_1 in particular is equal to $i\Theta = \Theta i$ and that its square is zero. Moreover it anticommutes with the de Rham differential d. Hence one actually has a *mixed complex* $(\Omega(LM), d, p_1)$. This observation will be important later.

We now pass to the iterated integrals. Notice first the shifting of the degree between forms on M and forms on LM by the following example: if w is a 1-form on M, then $\int_c w$ is a function on LM, where $c \in LM$. Iterated integrals generalise this idea. If w is a form on M, let $w(t)$ be the form $e_t^*(w)$ on LM, namely the pull back of w via the evaluation map $e_t : LM \to M$ given by evaluating loops at time t. Given forms $w_0, w_1, ..., w_k$ on M, the iterated integral

$$\sigma(w_0, ..., w_k)$$

is a form on LM of total degree $|w_0| + ... + |w_k| - k$ defined by the formula

$$\sigma(w_o, ..., w_k) = \int_{\Delta_k} w_0(0) \wedge i w_1(t_1) \wedge ... \wedge i w_k(t_k) dt_1 ... dt_k$$

where Δ_k is the standard k-simplex and i is the contraction operator with the tangent vector to the loop.

We can rewrite the above formula for the iterated integral using the maps p_k, namely

$$\sigma(w_0, ..., w_k) = w_0 p_k(w_1(0), ..., w_k(0))$$

One can then build a model for the forms on LM using iterated integrals. We begin by recalling the definition of the **cyclic bar complex** [1] of the algebra $\Omega(M)$. Let $C(\Omega(M))$ be the direct sum

$$\sum_{k=0}^{\infty} \Omega(M) \otimes s\Omega(M)^{\otimes k}$$

Here s is the suspension functor on graded vector spaces, that is the functor which simply reduces degree by 1. In general, the cyclic bar complex of any associative algebra comes naturally equipped with two "differentials", the *Hochschild*

differential b_0 defined via

$$b_0(w_0,...,w_k) = -\sum_{i=0}^{k-1}(-1)^{\epsilon_i}(w_0,...,w_{i-1},w_iw_{i+1},w_{i+2},...,w_k)$$
$$+(-1)^{(|w_k|-1)\epsilon_{k-1}}(w_kw_0,w_1,...,w_{k-1})$$

and *Connes'* differential B defined via

$$B(w_0,...,w_k) = \sum_{i=0}^{k}(-1)^{(\epsilon_{i-1}+1)(\epsilon_k-\epsilon_{i-1})}(1,w_i,...,w_k,w_0,...,w_{i-1})$$
$$-\sum_{i=0}^{k}(-1)^{(\epsilon_{i-1}+1)(\epsilon_k-\epsilon_{i-1})}(w_i,...,w_k,w_0,...,w_{i-1},1)$$

However, in our case we are interested in the cyclic bar complex of the algebra $\Omega(M)$ which is itself also a DGA with de Rham differential d. In the bar complex then one has an extension of this d, still denoted d, given via

$$d(w_0,...,w_k) = -\sum_{i=0}^{k}(-1)^{\epsilon_{i-1}}(w_0,...,w_{i-1},dw_i,w_{i+1},...,w_k)$$

As before, $\epsilon_i = |w_0| + ... + |w_i| - i$

Now we combine the de Rham differential d on $C(\Omega(M))$ with the Hochschild differential b_0 on $C(\Omega(M))$ to get a single differential b:

$$b = d + b_0$$

We shall refer to this cohomology $(C(\Omega(M)), b)$ as the *Hochschild cohomology* of $\Omega(M)$ Now one also has a *mixed complex* for the cyclic bar complex $C(\Omega(M))$, namely $(C(\Omega(M)), b, B)$. The total differential $b + B$ gives the *cyclic cohomology* of $\Omega(M)$

And now we are ready to state the main results relating iterated integrals, forms on loop space LM and the cyclic bar complex of the algebra of forms on M (see [6] for proofs and further explanations on notation and terminology):

Theorems :

1. The iterated integral map σ induces a map between the two mixed complexes

$$(C(\Omega(M)), b, B) \to (\Omega(LM), d, p_1)$$

This simply means that $p_1\sigma = \sigma B$ and $\sigma b = d\sigma$. (For the proof see [6]).

2. If M is simply connected, one has an isomorphism between the de Rham cohomology of the loop space and the Hochschild cohomology of the algebra of forms on M induced by the iterated integral map σ, namely

$$\sigma : (C(\Omega(M)), b) \to (\Omega(LM), d)$$

is an isomorphism in cohomology. (Again for the proof see [6] or [5]).

3. "Essentially" the cyclic cohomology of $\Omega(M)$ is isomorphic to the T-equivariant cohomology of LM. (The word "essentially" means that we ignore the complications that lead to the correct Jones' variant HC^{-}_{-*} functor). (For the proof see [7]).

4. Under the map σ, the shuffle product on the normalised cyclic bar complex $N(\Omega(M))$ of $\Omega(M)$ is carried into the wedge product on $\Omega(LM)$. (For the proof and terminology see [6]).

5. The forms on LM which are images of the iterated integral map are basic with respect to the action of the Lie pair ($vect[0,1], Diff[0,1]$). In particular this means that they are reparametrisation invariant *under reparametrisations of the loops* [6].

We just want to end this section by mentioning that many of the above generalise to actions of arbitrary compact Lie groups G acting on manifolds. One then gets *equivariant* versions of the above results. This is actually what we need in our physical problem, because we consider gauge theories and *Lie algebra valued* forms whereas all the above discussion refers to *real* valued forms. Fortunately, if one starts with a principal bundle P over a base manifold M with structure group G, the loop space LP also is a principal G-bundle over LM in a natural way, see [3]. This actually implies that the generalisations are straightforward.

0.5 The Proof

To begin with, in physics people usually work with *based* loops, which means that one picks a point $x \in M$ and considers loops having this point as starting and ending point. We shall denote this space $LM(x)$. The reason for this is that one can compose loops easily in this way.

This is not actually important in our treatment because we shall use formulae valid for the bigger space of free loops. We must however mention that here we consider smooth loops whereas in physics one can consider more general loops which are only continuous.

First one writes the holonomy element h as an infinite sum of iterated integrals by expanding the Dyson ordering. We use the formula of the σ map in terms of

the maps p_k where we conventionally define $p_0 = 1$ and we assume the 0th form $w_0(0)$ appearing in the formula to be equal to the constant form 1 and thus we omit it, since there is no integration on that either.

In more concrete terms then

$$h = \sum_{n=o}^{\infty} p_n(A_1, ..., A_n)$$

where we simplify the notation slightly by writing A_i instead of $A_i(0)$, $1 \leq i \leq n$. In our notation the index i states the "*position*" of the form A. Recall that because this is the holonomy element, namely an element of the structure group G, we know that the above converges.

Anyway, then one goes on by looking at Polyakov's variational formula and it is not hard to suspect that this "looks like" taking the d of the holonomy element h. The d enters the sum and hits every individual term p_n. The interchange of d and $\sum$ is justified because both sides make sense. In fact by definition the d of the p_n is actually a sum of p_n's, applied to different forms (see formula of Proposition mentioned above). Moreover recall that as mentioned in [6], the sum $\sum_{n=0}^{\infty} p_n$ is a Hochschild cocycle no matter what forms it is applied to and also recall that each iterated integral is finite, hence the sum is well defined (converges):

$$dh = d\sum_{n=0}^{\infty} p_n(A_1, ..., A_n) =$$

$$= \sum_{n=o}^{\infty} dp_n(A_1, ..., A_n) =$$

We now take each term separately and applying the formula of Proposition above we get:

$$dp_0 = 0$$

$$dp_1(A_1) = p_1(dA_1)$$

$$dp_2(A_1, A_2) = p_2(dA_1, A_2) + p_2(A_1, dA_2) - p_1(A_1 \wedge A_2)$$

$$dp_3(A_1, A_2, A_3) = p_3(dA_1, A_2, A_3) + p_3(A_1, dA_2, A_3)$$
$$+p_3(A_1, A_2, dA_3) - p_2(A_1 \wedge A_2, A_3) - p_2(A_1, A_2 \wedge A_3)$$

$$+...$$

Introducing the curvature 2-form F of the connection 1-form A to be $F = dA - A \wedge A$, we have the folowing formula:

$$dh = p_1(F) +$$
$$+ p_2(F_1, A_2) + p_2(A_1, F_2) +$$
$$+ p_3(F_1, A_2, A_3) + p_3(A_1, F_2, A_3) + p_3(A_1, A_2, F_3) +$$
$$+...$$

$$= \sum_{k=1}^{\infty} (\sum_{j=1}^{k} p_k(A_1, ..., A_{j-1}, F_j, A_{j+1}, ..., A_k))$$

We shall now rewrite Polyakov's formula using iterated integrals and we shall see that it coincides with the above expression.

The key point in Polyakov's formula is that he actually "breaks" the loop at a point s and then integrates over all s, namely from 0 to 1. We know how to write the path order exponent using iterated integrals. However one now must distinguish between two simplices because we have broken the loop.

In all of our discussion above, the integrals were taken over the standard simlpices over the interval [0,1], namely Δ_1 is the interval [0,1] and then Δ_k was $\{(s_1, ..., s_k) \in \mathbf{R}^k | 0 \leq s_1 \leq ... \leq s_k \leq 1\}$. We shall continue to keep this notation for the simplices over the interval [0,1]. The same holds for the maps p_k.

Now we brake the loop at the point s, so we must, in addition, have two extra classes of simplices:

I. One will be denoted Δ_k^s and the Δ_1^s will simply be the interval [0,s] and the general Δ_k^s will be

$$\{(s_1, ..., s_k) \in \mathbf{R}^k | 0 \leq s_1 \leq ... \leq s_k \leq s\}$$

The corresponding maps p_k will be accordingly denoted p_k^s.

II. The other will be denoted Δ_k^1 and the Δ_1^1 will simply be the interval [s,1] and the general Δ_k^1 will be

$$\{(s_1, ..., s_k) \in \mathbf{R}^k | s \leq s_1 \leq ... \leq s_k \leq 1\}$$

The maps will be denoted p_k^1 in this case.

Now with the above notation, Polyakov's variation can be written as:

$$\delta h = \int_0^1 ds(\sum_{n=0}^{\infty} p_n^s(A_1, ..., A_n))iF(s)(\sum_{n=0}^{\infty} p_n^1(A_1, ..., A_n)$$

where we supress the indices and remember that the integral ds refers to the F factor, indicated as $F(s)$.

If we expand the above, we get:

$$\delta h = \int_0^1 ds(1 + p_1^s(A_1) + p_2^s(A_1, A_2) + ...)iF(s)\times$$

$$\times(1 + p_1^1(A_1) + p_2^1(A_1, A_2) + ...) =$$

$$= \int_0^1 ds(iF(s) + iF(s)p_1^1(A_1) + iF(s)p_2^1(A_1, A_2) + ...$$

$$+p_1^s(A_1)iF(s) + p_1^s(A_1)iF(s)p_1^1(A_1) + p_1^s(A_1)iF(s)p_2^1(A_1, A_2) + ...$$

$$+p_2^s(A_1, A_2)iF(s) + p_2^s(A_1, A_2)iF(s)p_1^1(A_1) + p_2^s(A_1, A_2)iF(s)p_2^1(A_1, A_2)$$

$$+...)$$

Now here comes algebraic topology to say that, in our notation

$$\int_0^1 ds\Delta_i^s * \Delta_j^1 = \Delta_{i+j+1}$$

where the extra vertex s goes in the $(i+1)$-st slot.

With the above in mind, the formula gives exactly our expression for dh, namely:

$$\delta h = p_1(F_1) + p_2(F_1, A_2) + p_3(F_1, A_2, A_3) + ...$$

$$+p_2(A_1, F_2) + p_3(A_1, F_2, A_3) + p_4(A_1, F_2, A_3, A_4) + ...$$

$$+p_3(A_1, A_2, F_3) + p_4(A_1, A_2, F_3, A_4) + p_5(A_1, A_2, F_3, A_4, A_5) + ...$$

$$+...$$

$$= \sum_{k=1}^{\infty}(\sum_{j=1}^{k} p_k(A_1, ..., A_{j-1}, F_j, A_{j+1}, ..., A_k)) = dh$$

QED

0.6 Remarks:

1. The appearence of the curvature F is natural: the reason is Theorem 2 above: the curvature has two terms, the dA term comes from the d part and the $A \wedge A$ part comes from the b_0 part of the Hochschild differential $b = d + b_0$. And it is the Hochschild cohomology of the cyclic bar complex which is isomorphic with the de Rham cohomology of the loop space.

2. With our formalism, the flat connection on the loop space (cf [9], [4]) is just

$$\tilde{A} = h^{-1} dh$$

where h^{-1} is expressed exactly like h using sum of iterated integrals but now the vertices of the simplices will be in the opposite order, namely

$$0 \leq s_n \leq ... \leq s_1 \leq 1$$

One can see that this is exactly the standard expression for flat connections for finite dimensional bundles. The proof that the above defined connection is flat now becomes trivial, exactly like the finite dimensional case.

3. Similarly the analogue of Yang-Mills equations for loop space simplifies drastically.

4. Finally, in virtue of Theorem 5 quoted above all the expresions are reparametrisation invariant since we use iterated integrals.

Bibliography

[1] Adams, J.F.: On the cobar construction. Proc. Nat. Acad. Sci. USA **42**, 409-412 (1956)

[2] Atiyah, M.F.: Circular symmetry and stationary phase approximation. Asterisque **131**, 43-59 (1985)

[3] Bismut, J-M.: Index theorem and equivariant cohomology on the loop space. Commun. Math. Phys. **98**, 213-237 (1985)

[4] Chan, H-M., Faridani, J., Tsou, S.T.: A Non-Abelian Yang-Mills analogue of classical electromegnetic duality. Phys. Rev. **D52**, 6134 (1995)

Chan, H-M., Faridani, J., Tsou, S.T.: A generalised Duality Symmetry for Non-Abelian Yang-Mills fields. Phys. Rev. **D53**, 7293 (1996)

Chan, H-M., Scharbach, P., Tsou, S.T.: On the loop space formulation of gauge theories. Ann. Phys. **166**, 396-421 (1986)

Chan, H-M., Tsou, S.T.: Some Elementary Gauge Theory Concepts. World Scientific 1993

[5] Chen, K.T.: Iterated integrals of differential forms and loop space homology. Ann. Math. **97**, 217-246 (1973)

[6] Getzler, E., Jones, J.D.S., Petrack, S.: Differential forms on loop spaces and the cyclic bar complex. Topology **30** No3, 339-371 (1991)

[7] Jones, J.D.S.: Cyclic homology and equivariant homology. Invent. Math. **87**, 403-423 (1987)

[8] Loday, J-L.: Cyclic Homology. Springer 1992

[9] Polyakov, A.: Gauge fields as rings of glue. Nucl. Phys. **B164**, 171-188 (1980)

Polyakov, A.: Quark Confinement and Topology of Gauge Theories. Nucl. Phys. **B120**, 429-458 (1977)

Polyakov, A.: Compact Gauge Fields and the Infrared Catastrophe. Phys. Lett. **59B**, 82-84 (1975)

Polyakov, A.: String Representations and Hidden Symmetries for Gauge Fields. Phys. Lett. **82B**, 247-250 (1979)

Polyakov, A.: Confining Strings. hep-th/9607049, Princeton preprint (1996)

[10] Wilson, K.: Confinement of Quarks. Phys. Rev. **D10**, 2445-2459 (1974)

[11] Witten, E.: Supersymmetry and Morse theory. Jour. of Diff. Geometry **17**, 661-692 (1982)